Short-Step Chebyshev Impedance Transformers

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Abstract—Impedance transforming networks are described which consist of short lengths of relatively high impedance transmission line alternating with short lengths of relatively low impedance line. The sections of transmission line are all exactly the same length (except for corrections for fringing capacitances), and the lengths of the line sections are typically short compared to a quarter wavelength throughout the operating band of the transformer. Tables of designs are presented which give exactly Chebyshev transmission characteristics between resistive terminations having ratios ranging from 1.5 to 10, and for fractional bandwidths ranging from 0.10 to 1.20. These impedance-transforming networks should have application where very compact transmission-line or dielectric-layer impedance transformers are desired.

I. General

TABLES OF Chebyshev impedance-transformer designs of lumped-element, low-pass-filter form are presented in [1]. The structures used consist of a ladder configuration of series inductances alternating with shunt capacitances. The transmission characteristic typically gives a sizable mismatch at dc, followed at somewhat higher frequencies by a Chebyshev pass band where the attenuation is very low, and at the upper edge of the pass band, the transmission characteristic cuts off in a manner typical of low-pass filters. It was also noted in [1] that by techniques such as are used for designing semi-lumped-element microwave filters [2], [3], it is possible to design semi-lumped-element impedance-transformers from the tables in [1]. Such procedures are approximate but can give very good results if the design is carefully worked out. Microwave semi-lumped-element transformers of this type give performance comparable to that of quarter-wave transformers, but can be much smaller in size.

In this present discussion, the design of semi-lumped-element impedance transformers consisting of short sections of transmission line is treated on an exact basis. Figure 1 shows a six-section transformer of the type under consideration. In this particular case the structure is of coaxial form, and it consists of a cascade of line sections having various impedances, each line section having the same effective length l. Such a transformer will give a good match between the line impedances Z_0 and Z_7 over a band of frequencies as indicated by the transducer attenuation characteristic shown in Fig. 2. The length l of the line sections in Fig. 1 is considerably less than $\lambda_m/4$, where λ_m is the wavelength at the mid-band frequency. This is also evident in Fig. 2, where it will be seen that at the midband frequency of the lowest pass band, the electrical length θ_m of the line sections is considerably less than $\pi/2$. Unlike conventional quarter-wave transformers, this type of impedance transformer has maximum attenuation when the line sections are a quarter-wavelength long i.e., when $\theta=\pi/2$). The attenuation characteristic in Fig. 2 is periodic and has a maximum attenuation of L_{A1}. This differs from the lumped-element case discussed in [1], where the attenuation characteristic is not periodic and the attenuation approaches infinity at high frequencies.

An interesting situation occurs when the length l of the line sections is chosen to be $\lambda_m/8$. In that case, θ_m is $\pi/4$, and for example, in Fig. 1, $Z_1=Z_2$, $Z_3=Z_4$, $Z_5=Z_6$. The significance of this is that under this condition a Chebyshev short-step transformer becomes identical with a conventional Chebyshev quarter-wave transformer having half as many line sections. Thus, in order to have the up and down variation of line impedances as suggested in Fig. 1, l must be less than $\lambda_m/8$. The tables presented herein are for the case where $l=\lambda_m/16$, and results are given elsewhere for the case where $l=\lambda_m/32$ [15]. Tables for quarter-wave transformers (which as explained above are equivalent to the case of $l=\lambda_m/8$ herein) will be found in [4].

II. Response Parameters

Let us now define various parameters for the response characteristic in Fig. 2. Electrical length θ will be used as a frequency variable, where θ is the electrical length of each individual line section of the short-step transformer. The center frequency of the primary pass band is defined by

$$\theta_m = \frac{\theta_a + \theta_b}{2} = \frac{2\pi l}{\lambda_m}, \tag{1}$$

and the fractional bandwidth of the pass band is given by

$$w = \frac{\theta_b - \theta_a}{\theta_m} \tag{2}$$

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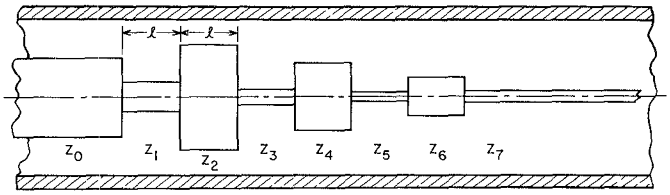

Fig. 1. A coaxial short-step impedance transformer designed to match between lines of impedance Z_0 and Z_7. The sections of impedance Z_1 to Z_6 are all of the same effective length l.

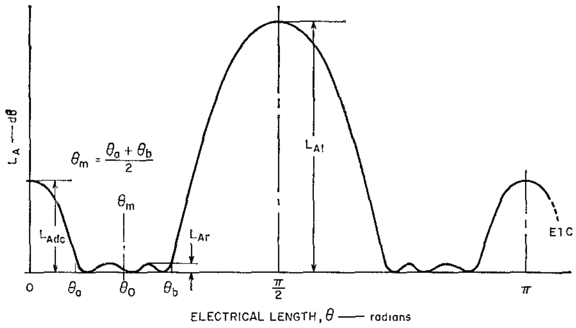

Fig. 2. The response characteristics of a typical short-step transformer. The parameter θ is the electrical length of each of the line sections (see Fig. 1).

where θ_a and θ_b are defined in Fig. 2. The attenuation L_{Adc} at dc can be computed by

$$L_{Adc} = 10 \log_{10} \frac{(r+1)^2}{4r} \text{ dB} \tag{3}$$

where r is the ratio of the terminating line impedance Z_{n+1} to the input line impedance Z_0.

Normalizing the line impedances so that $Z_0 = 1$ and $Z_{n+1} = r$, the peak attenuation L_{A1} can be computed by the formula

$$L_{A1} = 10 \log_{10} \frac{(1+R_{in})^2}{4R_{in}} \text{ dB} \tag{4}$$

where for $n/2$ even

$$R_{in} = \frac{r(Z_1 Z_3 \cdots Z_{n/2-1})^4}{(Z_2 Z_4 \cdots Z_{n/2})^4} \tag{5}$$

and for $n/2$ odd

$$R_{in} = \frac{(Z_1 Z_3 \cdots Z_{n/2})^4}{r(Z_2 Z_4 \cdots Z_{n/2-1})^4} \cdot \tag{6}$$

Values for the normalized impedances Z_1 to $Z_{n/2}$ are listed in the tables accompanying this discussion.

Another important parameter of the response in Fig. 2 is the pass band attenuation ripple L_{Ar}. Values of L_{Ar} are tabulated in Tables I through V for the various designs which are tabulated in Tables VI through X. These data are for the case where the line sections in Fig. 1 are $l = \lambda_m/16$ in length. The Tables of L_{Ar} values are of primary importance in determining which design should be used for a given application. Assuming that $l = \lambda_m/16$ is desired, for the desired termination ratio r and fractional bandwidth w, the corresponding value of L_{Ar} for the $n = 2$ section case in Table I should be checked. If this value of ripple is too large, then the table for $n = 4$ should be checked to see if $n = 4$ will result in a sufficiently small value of pass band ripple. If $n = 4$ still gives too large a value of L_{Ar}, then $n = 6$ should be tried, etc. In this manner the tables of L_{Ar} vs. r and w are used to determine the number of line sections required in order to give a sufficiently good degree of impedance match across the desired operating frequency band.

The entire response of a given short-step transformer design can be computed by use of the mapping techniques discussed in Section V of this discussion. This may be of interest for cases where the regions of high attenuation in the response characteristic of a short-step transformer are to be used for the purpose of filtering out unwanted signals. The attenuation in the stop bands can be very large. For example, for the case where $l = \lambda_m/16$, if $n = 6$ and $r = 5$, the peak stop band attenuation L_{A1} runs about 45 dB for the entire range of w covered by the tables. If the line-section lengths are $l = \lambda_m/32$, $n = 6$, and $r = 5$, then the peak attenuation L_{A1} runs about 83 dB for all the values of w covered by the tables.

TABLE I

L_{Ar} VS. r AND w FOR $n=2$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.0016	0.0064	0.0141	0.0243	0.0502	0.0798	0.1088	0.1342
2.0	0.0049	0.0191	0.0421	0.0724	0.1490	0.2352	0.3187	0.3906
2.5	0.0087	0.0344	0.0755	0.1295	0.2645	0.4146	0.5578	0.6797
3.0	0.0129	0.0509	0.1113	0.1905	0.3864	0.6009	0.8028	0.9725
4.0	0.0218	0.0855	0.1862	0.3168	0.6334	0.9705	1.2795	1.5336
5.0	0.0310	0.1211	0.2625	0.4439	0.8754	1.3230	1.7239	2.0475
6.0	0.0403	0.1570	0.3388	0.5695	1.1085	1.6545	2.1338	2.5146
7.0	0.0497	0.1930	0.4145	0.6928	1.3319	1.9655	2.5119	2.9403
8.0	0.0591	0.2289	0.4894	0.8134	1.5456	2.2573	2.8616	3.3302
9.0	0.0686	0.2647	0.5632	0.9312	1.7501	2.5319	3.1866	3.6894
10.0	0.0780	0.3003	0.6361	1.0462	1.9459	2.7908	3.4898	4.0220

TABLE II

L_{Ar} VS. r AND w FOR $n=4$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.0000	0.0001	0.0003	0.0009	0.0048	0.0148	0.0346	0.0655
2.0	0.0000	0.0002	0.0009	0.0028	0.0143	0.0443	0.1029	0.1935
2.5	0.0000	0.0003	0.0016	0.0051	0.0256	0.0794	0.1835	0.3423
3.0	0.0000	0.0005	0.0024	0.0075	0.0379	0.1171	0.2691	0.4980
4.0	0.0000	0.0008	0.0040	0.0127	0.0638	0.1958	0.4449	0.8100
5.0	0.0001	0.0011	0.0057	0.0180	0.0905	0.2758	0.6199	1.1113
6.0	0.0001	0.0015	0.0074	0.0235	0.1174	0.3558	0.7910	1.3978
7.0	0.0001	0.0018	0.0092	0.0289	0.1445	0.4351	0.9572	1.6692
8.0	0.0001	0.0022	0.0109	0.0344	0.1716	0.5135	1.1182	1.9261
9.0	0.0002	0.0025	0.0127	0.0400	0.1986	0.5908	1.2741	2.1696
10.0	0.0002	0.0028	0.0144	0.0455	0.2255	0.6669	1.4250	2.4008

TABLE III

L_{Ar} VS. r AND w FOR $n=6$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.0000	0.0000	0.0000	0.0000	0.0004	0.0022	0.0087	0.0254
2.0	0.0000	0.0000	0.0000	0.0001	0.0012	0.0067	0.0259	0.0758
2.5	0.0000	0.0000	0.0000	0.0002	0.0021	0.0121	0.0465	0.1354
3.0	0.0000	0.0000	0.0000	0.0003	0.0031	0.0179	0.0688	0.1992
4.0	0.0000	0.0000	0.0001	0.0005	0.0053	0.0302	0.1154	0.3310
5.0	0.0000	0.0000	0.0001	0.0007	0.0075	0.0430	0.1632	0.4635
6.0	0.0000	0.0000	0.0002	0.0009	0.0098	0.0558	0.2113	0.5943
7.0	0.0000	0.0000	0.0002	0.0011	0.0121	0.0688	0.2594	0.7225
8.0	0.0000	0.0000	0.0002	0.0013	0.0144	0.0818	0.3072	0.8478
9.0	0.0000	0.0000	0.0003	0.0015	0.0168	0.0949	0.3547	0.9700
10.0	0.0000	0.0000	0.0003	0.0017	0.0191	0.1079	0.4018	1.0892

TABLE IV

L_{Ar} VS. r AND w FOR $n=8$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0020	0.0090
2.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0010	0.0061	0.0269
2.5	0.0000	0.0000	0.0000	0.0000	0.0002	0.0018	0.0110	0.0483
3.0	0.0000	0.0000	0.0000	0.0000	0.0003	0.0026	0.0163	0.0713
4.0	0.0000	0.0000	0.0000	0.0000	0.0004	0.0045	0.0274	0.1197
5.0	0.0000	0.0000	0.0000	0.0000	0.0006	0.0063	0.0389	0.1692
6.0	0.0000	0.0000	0.0000	0.0000	0.0008	0.0083	0.0506	0.2191
7.0	0.0000	0.0000	0.0000	0.0000	0.0010	0.0102	0.0624	0.2689
8.0	0.0000	0.0000	0.0000	0.0000	0.0012	0.0121	0.0742	0.3184
9.0	0.0000	0.0000	0.0000	0.0001	0.0014	0.0141	0.0860	0.3675
10.0	0.0000	0.0000	0.0000	0.0001	0.0016	0.0160	0.0978	0.4162

TABLE V

L_{Ar} VS. r AND w FOR $n=10$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0031
2.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0014	0.0092
2.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0025	0.0165
3.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0038	0.0245
4.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0064	0.0413
5.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0009	0.0090	0.0586
6.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0012	0.0118	0.0761
7.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0015	0.0145	0.0937
8.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0018	0.0173	0.1114
9.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0021	0.0201	0.1291
10.0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0023	0.0229	0.1467

TABLE VI

Z_1 VS. r AND w FOR $n=2$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	1.8317	1.8271	1.8196	1.8094	1.7814	1.7452	1.7033	1.6579
2.0	2.4782	2.4709	2.4590	2.4426	2.3978	2.3393	2.2707	2.1956
2.5	3.0002	2.9911	2.9760	2.9552	2.8982	2.8238	2.7361	2.6396
3.0	3.4473	3.4366	3.4189	3.3946	3.3277	3.2403	3.1371	3.0233
4.0	4.2040	4.1907	4.1687	4.1385	4.0555	3.9468	3.8183	3.6761
5.0	4.8452	4.8297	4.8042	4.7692	4.6728	4.5463	4.3968	4.2312
6.0	5.4114	5.3941	5.3655	5.3262	5.2180	5.0761	4.9082	4.7222
7.0	5.9240	5.9050	5.8736	5.8304	5.7117	5.5559	5.3715	5.1670
8.0	6.3958	6.3752	6.3412	6.2945	6.1661	5.9975	5.7980	5.5766
9.0	6.8351	6.8131	6.7767	6.7268	6.5893	6.4089	6.1953	5.9583
10.0	7.2479	7.2245	7.1860	7.1329	6.9870	6.7954	6.5686	6.3170

TABLE VII-A

Z_1 VS. r AND w FOR $n=4$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	1.7831	1.7864	1.7918	1.7993	1.8194	1.8440	1.8686	1.8864
2.0	2.1446	2.1508	2.1610	2.1752	2.2142	2.2643	2.3185	2.3658
2.5	2.4008	2.4097	2.4244	2.4448	2.5016	2.5759	2.6589	2.7356
3.0	2.6037	2.6151	2.6340	2.6603	2.7340	2.8316	2.9425	3.0477
4.0	2.9205	2.9366	2.9633	3.0006	3.1062	3.2481	3.4121	3.5717
5.0	3.1679	3.1883	3.2223	3.2700	3.4056	3.5893	3.8035	4.0142
6.0	3.3731	3.3976	3.4386	3.4961	3.6603	3.8841	4.1462	4.4049
7.0	3.5498	3.5783	3.6258	3.6928	3.8846	4.1469	4.4547	4.7589
8.0	3.7058	3.7379	3.7919	3.8679	4.0864	4.3860	4.7379	5.0853
9.0	3.8458	3.8816	3.9418	4.0266	4.2711	4.6069	5.0013	5.3898
10.0	3.9732	4.0126	4.0788	4.1722	4.4420	4.8131	5.2485	5.6764

TABLE VII-B

Z_2 VS. r AND w FOR $n=4$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.5285	0.5295	0.5312	0.5337	0.5410	0.5523	0.5682	0.5895
2.0	0.5199	0.5214	0.5240	0.5277	0.5386	0.5547	0.5768	0.6049
2.5	0.5315	0.5335	0.5369	0.5416	0.5557	0.5763	0.6041	0.6388
3.0	0.5470	0.5494	0.5534	0.5592	0.5762	0.6011	0.6343	0.6753
4.0	0.5785	0.5817	0.5871	0.5947	0.6173	0.6502	0.6936	0.7463
5.0	0.6076	0.6115	0.6182	0.6276	0.6554	0.6957	0.7486	0.8121
6.0	0.6339	0.6385	0.6463	0.6574	0.6902	0.7377	0.7996	0.8732
7.0	0.6578	0.6631	0.6720	0.6847	0.7222	0.7766	0.8471	0.9302
8.0	0.6797	0.6856	0.6956	0.7099	0.7520	0.8130	0.8917	0.9838
9.0	0.6998	0.7064	0.7174	0.7332	0.7798	0.8472	0.9339	1.0346
10.0	0.7185	0.7257	0.7378	0.7550	0.8060	0.8797	0.9740	1.0829

TABLE VIII-A

Z_1 VS. r AND w FOR $n=6$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	1.5814	1.5851	1.5915	1.6006	1.6273	1.6663	1.7184	1.7817
2.0	1.7916	1.7977	1.8080	1.8227	1.8662	1.9312	2.0199	2.1317
2.5	1.9334	1.9414	1.9548	1.9740	2.0315	2.1180	2.2384	2.3931
3.0	2.0419	2.0514	2.0676	2.0907	2.1602	2.2660	2.4148	2.6090
4.0	2.2051	2.2173	2.2381	2.2680	2.3584	2.4978	2.6974	2.9633
5.0	2.3279	2.3424	2.3671	2.4027	2.5112	2.6801	2.9253	3.2564
6.0	2.4271	2.4436	2.4718	2.5125	2.6371	2.8329	3.1200	3.5118
7.0	2.5106	2.5290	2.5603	2.6057	2.7450	2.9657	3.2922	3.7410
8.0	2.5831	2.6031	2.6374	2.6870	2.8401	3.0841	3.4478	3.9509
9.0	2.6471	2.6688	2.7057	2.7593	2.9253	3.1915	3.5908	4.1459
10.0	2.7047	2.7278	2.7673	2.8246	3.0029	3.2902	3.7238	4.3287

TABLE VIII-B

Z_2 VS. r AND w FOR $n=6$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.5026	0.5030	0.5036	0.5046	0.5078	0.5133	0.5222	0.5361
2.0	0.4888	0.4896	0.4909	0.4928	0.4987	0.5086	0.5240	0.5471
2.5	0.4897	0.4908	0.4926	0.4952	0.5035	0.5171	0.5382	0.5694
3.0	0.4940	0.4953	0.4976	0.5009	0.5112	0.5281	0.5543	0.5930
4.0	0.5049	0.5067	0.5097	0.5141	0.5280	0.5508	0.5862	0.6384
5.0	0.5157	0.5179	0.5216	0.5269	0.5439	0.5720	0.6157	0.6802
6.0	0.5257	0.5282	0.5325	0.5387	0.5585	0.5913	0.6427	0.7188
7.0	0.5348	0.5376	0.5424	0.5495	0.5719	0.6091	0.6676	0.7546
8.0	0.5432	0.5463	0.5516	0.5594	0.5842	0.6255	0.6909	0.7882
9.0	0.5509	0.5542	0.5600	0.5685	0.5955	0.6409	0.7128	0.8200
10.0	0.5580	0.5616	0.5678	0.5770	0.6062	0.6553	0.7335	0.8502

TABLE VIII-C

Z_3 VS. r AND w FOR $n=6$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	3.1624	3.1573	3.1488	3.1371	3.1045	3.0604	3.0054	2.9390
2.0	3.8392	3.8321	3.8203	3.8041	3.7594	3.7001	3.6284	3.5450
2.5	4.3650	4.3565	4.3424	4.3231	4.2700	4.2006	4.1182	4.0245
3.0	4.8148	4.8051	4.7892	4.7674	4.7078	4.6307	4.5404	4.4395
4.0	5.5807	5.5692	5.5504	5.5247	5.4550	5.3662	5.2645	5.1541
5.0	5.2346	6.2217	6.2006	6.1718	6.0942	5.9965	5.8869	5.7703
6.0	6.8150	6.8009	6.7778	6.7464	6.6622	6.5574	6.4417	6.3212
7.0	7.3422	7.3270	7.3022	7.2685	7.1786	7.0678	6.9474	6.8242
8.0	7.8285	7.8124	7.7860	7.7502	7.6553	7.5393	7.4151	7.2901
9.0	8.2821	8.2651	8.2373	8.1997	8.1002	7.9796	7.8524	7.7264
10.0	8.7088	8.6910	8.6618	8.6225	8.5188	8.3942	8.2646	8.1379

TABLE IX-A

Z_1 VS. r AND w FOR $n=8$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	1.4220	1.4256	1.4318	1.4405	1.4666	1.5063	1.5627	1.6398
2.0	1.5549	1.5603	1.5694	1.5825	1.6218	1.6824	1.7705	1.8945
2.5	1.6419	1.6486	1.6599	1.6762	1.7256	1.8024	1.9158	2.0785
3.0	1.7070	1.7148	1.7280	1.7469	1.8046	1.8950	2.0301	2.2269
4.0	1.8031	1.8126	1.8287	1.8520	1.9233	2.0363	2.2081	2.4644
5.0	1.8740	1.8848	1.9034	1.9301	2.0125	2.1444	2.3475	2.6559
6.0	1.9304	1.9424	1.9630	1.9927	2.0847	2.2330	2.4640	2.8197
7.0	1.9774	1.9905	2.0128	2.0452	2.1456	2.3086	2.5651	2.9646
8.0	2.0177	2.0318	2.0557	2.0905	2.1986	2.3751	2.6551	3.0957
9.0	2.0532	2.0680	2.0935	2.1304	2.2456	2.4346	2.7367	3.2163
10.0	2.0848	2.1005	2.1272	2.1662	2.2879	2.4887	2.8116	3.3286

TABLE IX-B

Z_2 VS. r AND w FOR $n=8$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.5215	0.5213	0.5210	0.5207	0.5200	0.5201	0.5223	0.5287
2.0	0.5003	0.5003	0.5005	0.5007	0.5021	0.5056	0.5131	0.5280
2.5	0.4938	0.4941	0.4946	0.4954	0.4984	0.5046	0.5166	0.5386
3.0	0.4918	0.4922	0.4930	0.4942	0.4985	0.5071	0.5229	0.5513
4.0	0.4919	0.4926	0.4939	0.4958	0.5025	0.5149	0.5374	0.5769
5.0	0.4940	0.4949	0.4966	0.4991	0.5077	0.5235	0.5517	0.6010
6.0	0.4966	0.4977	0.4997	0.5027	0.5130	0.5317	0.5650	0.6233
7.0	0.4992	0.5006	0.5029	0.5064	0.5180	0.5394	0.5774	0.6440
8.0	0.5019	0.5034	0.5060	0.5098	0.5229	0.5466	0.5889	0.6634
9.0	0.5044	0.5061	0.5089	0.5132	0.5274	0.5534	0.5998	0.6817
10.0	0.5069	0.5086	0.5117	0.5163	0.5317	0.5598	0.6101	0.6991

TABLE IX-C

Z_3 VS. r AND w FOR $n=8$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	2.9464	2.9450	2.9426	2.9391	2.9289	2.9143	2.8954	2.8725
2.0	3.3791	3.3781	3.3764	3.3739	3.3665	3.3561	3.3440	3.3327
2.5	3.6898	3.6895	3.6890	3.6882	3.6856	3.6823	3.6807	3.6855
3.0	3.9416	3.9421	3.9430	3.9441	3.9470	3.9518	3.9619	3.9840
4.0	4.3468	4.3491	4.3528	4.3579	4.3725	4.3943	4.4287	4.4865
5.0	4.6739	4.6779	4.6845	4.6937	4.7199	4.7587	4.8175	4.9107
6.0	4.9524	4.9581	4.9676	4.9807	5.0183	5.0737	5.1564	5.2845
7.0	5.1972	5.2045	5.2168	5.2338	5.2825	5.3541	5.4600	5.6222
8.0	5.4169	5.4259	5.4408	5.4616	5.5210	5.6084	5.7371	5.9326
9.0	5.6172	5 6277	5.6452	5.6696	5.7396	5.8424	5.9932	6.2212
10.0	5.8018	5.8139	5.8339	5.8619	5.9421	6.0598	6.2324	6.4922

TABLE IX-D

Z_4 VS. r AND w FOR $n=8$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.4096	0.4106	0.4123	0.4146	0.4212	0.4304	0.4423	0.4571
2.0	0.4373	0.4386	0.4408	0.4439	0.4527	0.4650	0.4810	0.5010
2.5	0.4662	0.4678	0.4705	0.4743	0.4850	0.5000	0.5195	0.5440
3.0	0.4933	0.4952	0.4983	0.5027	0.5151	0.5326	0.5553	0.5839
4.0	0.5416	0.5439	0.5478	0.5533	0.5689	0.5908	0.6194	0.6555
5.0	0.5836	0.5863	0.5909	0.5974	0.6158	0.6417	0.6756	0.7188
6.0	0.6208	0.6240	0.6293	0.6366	0.6577	0.6873	0.7262	0.7758
7.0	0.6545	0.6580	0.6639	0.6721	0.6957	0.7288	0.7724	0.8281
8.0	0.6853	0.6892	0.6957	0.7047	0.7306	0.7671	0.8151	0.8767
9.0	0.7138	0.7181	0.7251	0.7349	0.7630	0.8027	0.8550	0.9223
10.0	0.7404	0.7450	0.7525	0.7631	0.7934	0.8361	0.8925	0.9652

TABLE X-A

Z_1 VS. r AND w FOR $n=10$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1 5	1.3080	1.3112	1.3167	1.3246	1.3485	1.3854	1.4394	1.5173
2.0	1.3965	1.4012	1.4090	1.4202	1.4544	1.5078	1.5878	1.7062
2.5	1.4534	1.4589	1.4684	1.4820	1.5236	1.5893	1.6888	1.8390
3.0	1.4954	1.5017	1.5125	1.5280	1.5756	1.6511	1.7668	1.9439
4.0	1.5567	1.5642	1.5769	1.5954	1.6523	1.7437	1.8858	2.1083
5.0	1.6013	1.6097	1.6241	1.6449	1.7092	1.8133	1.9771	2.2379
6.0	1.6365	1.6457	1.6614	1.6842	1.7547	1.8696	2.0521	2.3469
7.0	1.6657	1.6755	1.6924	1.7168	1.7928	1.9171	2.1163	2.4419
8.0	1.6906	1.7010	1.7189	1.7448	1.8256	1.9585	2.1729	2.5269
9.0	1.7124	1.7234	1.7421	1.7694	1.8546	1.9952	2.2236	2.6044
10.0	1.7318	1.7432	1.7628	1.7913	1.8805	2.0283	2.2698	2.6757

TABLE X-B

Z_2 VS. r AND w FOR $n=10$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.5547	0.5539	0.5527	0.5511	0.5467	0.5416	0.5372	0.5355
2.0	0.5282	0.5277	0.5267	0.5255	0.5225	0.5199	0.5196	0.5251
2.5	0.5173	0.5168	0.5161	0.5152	0.5134	0.5127	0.5157	0.5270
3.0	0.5114	0.5110	0.5105	0.5099	0.5090	0.5100	0.5159	0.5322
4.0	0.5054	0.5053	0.5051	0.5050	0.5056	0.5093	0.5199	0.5446
5.0	0.5027	0.5027	0.5027	0.5030	0.5049	0.5108	0.5253	0.5573
6.0	0.5013	0.5014	0.5017	0.5023	0.5052	0.5130	0.5309	0.5694
7.0	0.5006	0.5008	0.5013	0.5021	0.5060	0.5154	0.5365	0.5808
8.0	0.5003	0.5006	0.5013	0.5024	0.5070	0.5180	0.5418	0.5916
9.0	0.5002	0.5006	0.5014	0.5027	0.5082	0.5205	0.5469	0.6018
10.0	0.5003	0.5008	0.5017	0.5032	0.5093	0.5230	0.5518	0.6115

TABLE X-C

Z_3 VS. r AND w FOR $n=10$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	2.6911	2.6925	2.6947	2.6978	2.7057	2.7152	2.7253	2.7362
2.0	2.9862	2.9887	2.9928	2.9985	3.0138	3.0339	3.0588	3.0914
2.5	3.1838	3.1874	3.1934	3.2016	3.2245	3.2557	3.2964	3.3522
3.0	3.3367	3.3414	3.3491	3.3599	3.3901	3.4322	3.4884	3.5671
4.0	3.5719	3.5785	3.5895	3.6049	3.6489	3.7113	3.7969	3.9193
5.0	3.7536	3.7620	3.7761	3.7957	3.8521	3.9332	4.0460	4.2094
6.0	3.9037	3.9137	3.9305	3.9540	4.0219	4.1204	4.2586	4.4605
7.0	4.0325	4.0440	4.0633	4.0905	4.1691	4.2838	4.4459	4.6845
8.0	4.1459	4.1589	4.1805	4.2111	4.2997	4.4297	4.6145	4.8882
9.0	4.2477	4.2619	4.2859	4.3196	4.4178	4.5623	4.7687	5.0760
10.0	4.3402	4.3557	4.3818	4.4185	4.5258	4.6841	4.9113	5.2510

TABLE X-D

Z_4 VS. r AND w FOR $n=10$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	0.3918	0.3927	0.3941	0.3962	0.4021	0.4106	0.4221	0.4369
2.0	0.4048	0.4060	0.4080	0.4108	0.4190	0.4309	0.4469	0.4679
2.5	0.4198	0.4212	0.4237	0.4272	0.4373	0.4520	0.4719	0.4982
3.0	0.4339	0.4356	0.4385	0.4425	0.4543	0.4715	0.4949	9.5260
4.0	0.4589	0.4610	0.4645	0.4696	0.4843	0.5058	0.5354	0.5752
5.0	0.4802	0.4826	0.4868	0.4927	0.5100	0.5354	0.5704	0.6180
6.0	0.4988	0.5016	0.5063	0.5129	0.5325	0.5614	0.6013	0.6561
7.0	0.5153	0.5184	0.5236	0.5310	0.5527	0.5847	0.6293	0.6907
8.0	0.5303	0.5337	0.5393	0.5473	0.5710	0.6060	0.6549	0.7226
9.0	0.5440	0.5476	0.5537	0.5623	0.5878	0.6256	0.6786	0.7523
10.0	0 5566	0.5605	0.5669	0.5762	0.6034	0.6438	0.7007	0.7801

TABLE X-E

Z_5 VS. r AND w FOR $n=10$ AND $l=\lambda_m/16$

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0	1.2
1.5	3.5808	3.5731	3.5604	3.5428	3.4941	3.4290	3.3504	3.2611
2.0	4.2050	4.1953	4.1794	4.1574	4.0966	4.0160	3.9196	3.8119
2.5	4.7135	4.7024	4.6842	4.6591	4.5897	4.4982	4.3894	4.2691
3.0	5.1576	5.1454	5.1253	5.0976	5.0213	4.9210	4.8023	4.6721
4.0	5.9249	5.9109	5.8878	5.8560	5.7686	5.6541	5.5198	5.3743
5.0	6.5868	6.5713	6.5457	6.5106	6.4141	6.2882	6.1414	5.9841
6.0	7.1774	7.1606	7.1329	7.0949	6.9906	6.8550	6.6976	6.5306
7.0	7.7155	7.6976	7.6680	7.6274	7.5162	7.3720	7.2054	7.0302
8.0	8.2130	8.1940	8.1627	8.1197	8.0023	7.8503	7.6755	7.4932
9.0	8.6776	8 6576	8.6248	8.5797	8.4565	8.2975	8.1152	7.9266
10.0	9.1151	9.0943	9.0599	9.0129	8.8844	8.7187	8.5296	8.3354

III. The Impedance Tables

Line impedances for designs having line section lengths $l=\lambda_m/16$ are presented in Tables VI to X. (Reference [15] gives tables for $l=\lambda/16$ and $l=\lambda/32$ for termination ratios $r=1.5$ to 20.) In all of these tables the impedance values are normalized so that

$$Z_0 = 1, \quad \text{and} \quad Z_{n+1} = r \tag{7}$$

for the terminating line impedances on the left and right, respectively. Since these circuits are antimetric, half of the line impedances can be computed from the other half. For this reason only half of the line impedances for each design are included in the tables. The remaining line impedances are easily computed by use of the formula

$$Z_j \Bigg|_{j=(n/2)+1 \text{ to } n} = \frac{r}{Z_{n+1-j}} \,. \tag{8}$$

It will be noted that there are tables for the cases of $n=2$, 4, 6, 8, and 10 for the case of $l=\lambda_m/16$. For each value of n there is a separate table for Z_1, a separate table for Z_2, etc., up to $Z_{n/2}$.

It is thus readily seen that a desired design is obtained by using the tables of L_{Ar} in order to determine the required value of n, and then selecting the impedance values for half of the circuit from the corresponding tables of impedances. The tables give the impedances for half of the network, and the remaining impedances are then obtained by use of (8).

IV. Corrections for Fringing Capacitances, and Modification of the Impedance Values

The structure shown in Fig. 1 (and equivalent structures in waveguide) will have fringing capacitances which occur at the steps between lines of different impedance. The fringing fields at these steps may be represented by inserting lumped capacitances wherever a junction between lines of different impedance occurs. Reference [5] or [7] gives data from which the fringing capacitance values for coaxial-line step discontinuities can be obtained, and [6] or [7] gives corresponding data for steps in waveguide. Having values for these fringing capacitances, the designer can compensate for them by making small adjustments in the physical lengths of the various line sections in the short-step impedance transformer. Such corrections are important to make if best performance is to be obtained.

One way[1] to compensate for the fringing capacitances is to replace the short sections of transmission lines in Fig. 1 by lumped-element equivalent circuits. Figure 3(a) shows a length of transmission line, and the circuit on the left in Fig. 3(b) shows an exact equivalent circuit for this transmission line. On the right in Fig. 3(b) is shown an approximate equivalent circuit for the transmission line which is valid if the line length l is small

[1] Another way is that given in [4], Sec. 6.08.

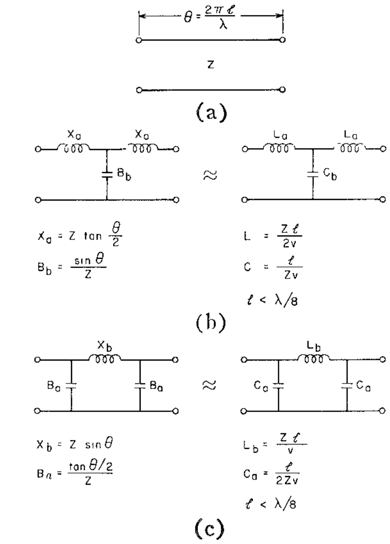

Fig. 3. Equivalent circuits for a length of transmission line. The parameter Z is the characteristic impedance of the line, l is the line length, and v is the velocity of propagation.

compared to a quarter wavelength (say, less than $\lambda/8$). Figure 3(c) shows another exact equivalent circuit for the length of transmission line, and on the right is shown yet another approximate equivalent circuit, which again is accurate if l is small compared to a quarter wavelength. For the structures under consideration, the length of the line sections in the operating frequency band of interest will be considerably less than a quarter wavelength, and the approximate T- and π-equivalent circuits on the right in Fig. 3 will be useful.

In Fig. 1 and in the other circuit designs tabulated herein, the even-numbered line sections operate predominantly like shunt capacitors. It is therefore convenient to lump the fringing capacitances in with these line sections. In order to compensate for the added capacitances due to the fringing fields, we may compute the capacitance C_a in the equivalent circuit on the right in Fig. 3(c), for each even-numbered line section. Then for each even-numbered line section we compute a compensated length l' by making use of the equation

$$C_{fL} + C_{fR} + \frac{l'}{Z_k v} = 2C_a \tag{9}$$

where C_{fL} is the fringing capacitance at the left end of the line section being treated, C_{fR} is the fringing capacitance at the right end, Z_k is the characteristic impedance of the line section, and v is the velocity of propagation. The third term on the left in (9) will be recognized as being the shunt capacitance of a length of transmission line of impedance Z_k and of length l'. Solving (9) for l' gives a compensated length l' which is slightly shorter than the original design length l.

To summarize, a recommended procedure for compensating for the fringing effects in circuits such as Fig. 1 is to determine the fringing-capacitance values using [5], [6], or [7], and then to solve for compensated lengths l' for the even-numbered line sections by solving (9).

(The capacitance of each line section, plus its fringing capacitance, will then equal the total desired capacitance for the given line section.) It appears reasonable to ignore the small changes in inductance that result from this procedure.

A special case occurs with respect to the fringing capacitance and that exists at the junction between lines Z_0 and Z_1 in Fig. 1. This fringing capacitance may also be compensated for by a somewhat different technique. Besides the approximate equivalent circuits shown on the right in Fig. 3, another reasonably accurate equivalent circuit for a short length of transmission line is an L-section circuit consisting of a series inductance L_a and a shunt capacitance C_a given by the values

$$L_a = \frac{Zd}{v} \tag{10}$$

$$C_a = \frac{d}{Zv} \tag{11}$$

where d is the length of the section of transmission line, Z is the characteristic impedance of the line, and v is again the velocity of propagation. Suppose the fringing capacitance at the junction between Z_0 and Z_1 is C_f. If a series inductance ΔL is placed beside this shunt-fringing capacitance C_f, an L-section circuit can be obtained having a characteristic impedance which will match the terminating-line-impedance Z_0. In terms of equations this gives

$$Z_0 = \sqrt{\frac{\Delta L}{C_f}} = \sqrt{\frac{L_a}{C_a}}\,. \tag{12}$$

Solving for ΔL gives

$$\Delta L = C_f Z_0^2. \tag{13}$$

A practical way to obtain the desired inductance ΔL is to increase the length of the transmission line Z_1 by an amount d, which has the desired amount of inductance. By (10), the inductance ΔL for a given value of d and line impedance Z_1 is given by

$$\Delta L = \frac{Z_1 d}{v}\,. \tag{14}$$

Solving (13) and (14) for d gives

$$d = \frac{v Z_0^2 C_f}{Z_1}\,. \tag{15}$$

Thus, in order to compensate for the fringing capacitance C_f between lines Z_0 and Z_1 in the structure in Fig. 1 (or in any of the other designs tabulated herein), the length of line section Z_1 should be *increased* from l to $l+d$, where d is computed by use of (15).

V. Theory by Which the Tables Were Obtained

In some respects the synthesis procedure used for obtaining the line-impedance values in the tables presented herein is similar to that discussed by Riblet [8] for the design of Chebyshev, quarter-wave step transformers. In discussing the synthesis procedure used herein, we will make use of terminology similar to that of Richards [9], Ozaki and Ishii [10], and Wenzel [11].

The transfer function for a step transformer such as that in Fig. 1 would involve transcendental functions. In order to eliminate such functions, and to simplify the synthesis problem, Richards [9] makes use of the mapping function

$$p = \tanh \frac{as}{2} \tag{16}$$

where $s = u + j\Omega$ is the complex-frequency variable for the transmission-line circuit, and $p = \sigma + j\omega$ is the frequency variable of the mapped transfer function. The parameter a used by Richards is defined by

$$\frac{a}{2} = \frac{l}{v} = \frac{\pi}{2\Omega_{1/4}} \tag{17}$$

where v is the velocity of propagation, and $\Omega_{1/4}$ is the radian frequency for which $l = \lambda_m/4$. In order to apply the mapping in (16), it is necessary that all of the transmission lines in the structure be of length l or of some integer multiple thereof. Using this mapping eliminates the transcendental nature of the transfer and impedance functions for transmission-line circuits, and makes possible the use of well-known network synthesis techniques which are commonly used for lumped-element circuits.

In terms of the characteristic impedance Z of a line section, s, and a, the open-circuit impedance parameters for a length of transmission line are given by

$$z_{11} = z_{22} = \frac{Z}{\tanh \frac{as}{2}} \tag{18}$$

$$z_{12} = z_{21} = \frac{Z}{\sinh \frac{as}{2}}\,. \tag{19}$$

After applying the mapping in (16), these open-circuit impedance parameters become

$$z_{11} = z_{22} = \frac{Z}{p} \tag{20}$$

$$z_{12} = z_{21} = \frac{Z\sqrt{1-p^2}}{p} = \frac{Z\sqrt{(1-p)(1+p)}}{p}\,. \tag{21}$$

These functions look like the open-circuit impedance functions for a lumped-element circuit in terms of a complex frequency variable p, except for the fact that the transfer impedance functions $z_{12}=z_{21}$ have one-half order zeros of transmission at $p=1$ and $p=-1$. Except for this square root, the open-circuit impedances in (20) and (21) would characterize a physically realizable lumped-element circuit. Because of the nonphysical realizability of the half-order zeros of transmission in this circuit, Richards [9] has named the p-plane equivalent of a transmission line, a "unit element" (abbreviated U.E.). Thus the p-plane equivalent of the transmission line circuit in Fig. 1 is the cascade of unit elements shown in Fig. 4.

Each of the unit elements in the circuit in Fig. 4 contributes to the transfer function, a half-order zero of transmission at $p=1$, and half-order zero of transmission at $p=-1$. Thus, for an n-section circuit of this sort, there will be n half-order zeros of transmission at $p=1$ and also at $p=-1$. In terms of an attenuation function, the zeros of transmission become poles of attenuation, and the transfer function for an n-section circuit of the sort in Fig. 4 will be of the form

$$\frac{E_0}{E_{n+1}} = \frac{b_n p^n + b_{n-1}p^{n-1} + \cdots b_3 p^3 + b_2 p^2 + b_1 p + b_0}{(\sqrt{1-p^2})^n} = \frac{U(p)}{(\sqrt{1-p^2})^n}. \tag{22}$$

When this transfer function is factored so as to display the locations of its poles and zeros, it takes the form

$$\frac{E_0}{E_{n+1}} = \frac{b_n(p-p_1)(p-p_2)\cdots(p-p_n)}{[j\sqrt{(p-1)(p+1)}]^n}. \tag{23}$$

By use of conventional lumped-element-circuit network synthesis techniques, along with some techniques in Richards' paper [9], it is possible to synthesize circuits of the form Fig. 4 so as to have a prescribed transfer function of the form in (22) and (23).

The next problem to be treated is the problem of synthesizing transfer functions of the form in (22) or (23) so as to have a Chebyshev pass band. Let us first investigate what the frequency response in the p plane should look like. For frequencies $s=j\Omega$, the transformation in (16) becomes

$$j\omega = j\tan\frac{a\Omega}{2}. \tag{24}$$

Dropping the j's and replacing $a\Omega/2$ by θ,

$$\omega = \tan\theta. \tag{25}$$

If we apply this mapping to the abscissa scale in Fig. 2, we will obtain a mapped transfer function of the form in Fig. 5, where the function is sketched also for negative frequencies $-\omega$, for reasons of clarifying some of the later discussion. Note that application of the mapping (25) has eliminated the periodic character which the transfer function had in Fig. 2.

The transfer function in Fig. 5 is in some respects similar to the transfer function for analogous cases discussed in [1], and we shall make use of some similar synthesis techniques. In [1], the desired transfer function was obtained by a mapping from the transfer function of a conventional Chebyshev lumped-element low-pass filter circuit. The transfer function for a conventional three-reactive-element Chebyshev low-pass filter circuit is shown in Fig. 6, where again the transfer function has been sketched also for negative frequencies. It will be noted that this transfer function has three ripple-minima when negative frequencies are included), while the transfer function in Fig. 5 has two pass bands and six minima when negative frequencies are included). Therefore, in order to obtain the transfer function in Fig. 5 from that in Fig. 6, the transfer func-

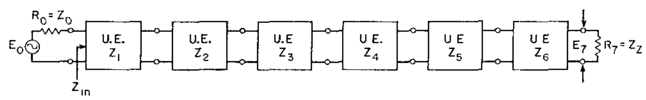

Fig. 4. p-plane equivalent circuit of the transmission-line circuit in Fig. 1.

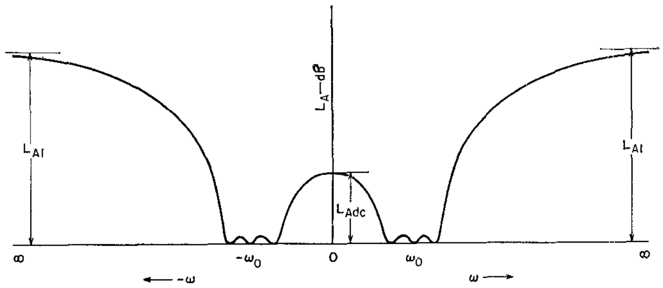

Fig. 5. The attenuation characteristic in Fig. 2 mapped to the p-plane by mapping $\omega=\tan\theta$.

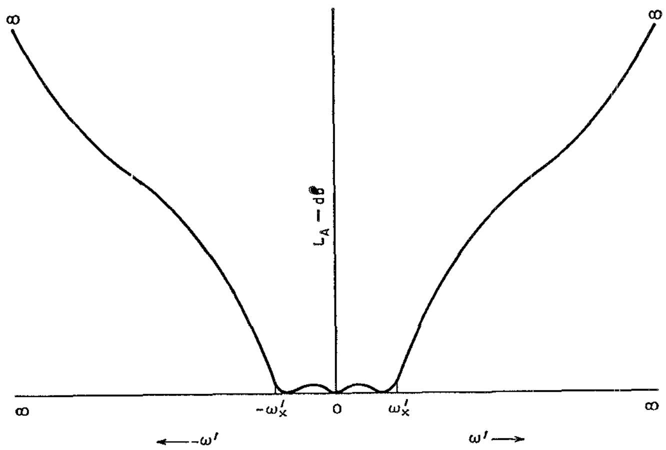

Fig. 6. Transducer attenuation of a conventional three-reactive-element Chebyshev low-pass filter. For mathematical purposes the characteristic for negative frequencies is also shown.

tion in Fig. 6 must map into that in Fig. 5 twice. Another requirement here is that the multiple-order poles of attenuation at infinity in the p' complex frequency plane for the transfer function in Fig. 6 must map to multiple-order poles at $p=1$ and $p=-1$ for the transfer function in Fig. 5, as is required by the denominators of (22) or (23). Some reflection on the matter will show that a mapping function which accomplishes this is

$$\omega' = \omega_x' A \left(\frac{\omega^2 - \omega_0^2}{\omega^2 + 1}\right), \tag{26}$$

where ω' is the sinusoidal frequency variable for the conventional low-pass filter, ω_x' is the cutoff frequency of the conventional filter, and ω is the frequency variable for the circuit of the form in Fig. 4. The parameter ω_0 is as defined in Fig. 5, and it will be noted that the point $\omega'=0$ in Fig. 6 maps to the points $\omega=\omega_0$ and $\omega=-\omega_0$ in Fig. 5. Extending the mapping in (26) to complex frequencies by inserting $p=j\omega$ and $p'=j\omega'$, we obtain

$$p' = j\omega_x' A \left(\frac{p^2 + \omega_0^2}{p^2 - 1}\right). \tag{27}$$

The voltage attenuation function for a conventional lumped-element, Chebyshev, low-pass filter circuit in terms of complex frequencies p' is of the form

$$\frac{E_0}{E_{q+1}} = c_q p'^q + c_{q-1} p'^{q-1} + \cdots + c_1 p' + c_0 = U'(p') \tag{28}$$

and

$$= c_q(p' - p_1')(p - p_2') \cdots (p' - p_q'). \tag{29}$$

The corresponding transducer power attenuation ratio function is of the form

$$\frac{P'_{\text{avail}}}{P_L'} = g' U'(p') U'(-p') \tag{30}$$

where P'_{avail} is the available power of the generator, P_L' is the power delivered to the load, $U'(p')$ is as defined in (28), and g' is a real, constant multiplier. For sinusoidal frequencies, the transducer attenuation in dB is given by

$$L_A = 10 \log_{10} \frac{P'_{\text{avail}}}{P'_{n+1}}\bigg|_{p'=j\omega'} \tag{31}$$

and it is this function which is sketched in Fig. 6. The corresponding transducer-power attenuation ratio function for the circuit in Fig. 4 is of the form

$$\frac{P_{\text{avail}}}{P_L} = \frac{gU(p)U(-p)}{(1 - p^2)^n} \cdot \tag{32}$$

It is readily seen that by inserting the mapping function in (27) into (30), a transducer attenuation ratio function of the desired form in (32) will be obtained. It should be noted that the power of the numerator polynominal in (32) will then be double the power of the numerator polynomial in (30). As a result, for the corresponding voltage attenuation ratio functions in (28) and (22), $n=2q$.

Closed-form expressions giving the locations of the zeros of transducer power attenuation ratio functions of the form in (30) for conventional Chebyshev low-pass filters are available [12]. However, it will be more convenient to work with the reflection coefficient function

$$\Gamma'(p') = \frac{N'(p')}{U'(p')} \cdot \tag{33}$$

Putting this function in power reflection coefficient form, we obtain

$$\frac{P_{\text{refl}}}{P_{\text{avail}}} = \Gamma'(p')\Gamma'(-p') = \frac{N'(p')N'(-p')}{U'(p')U'(-p')} \tag{34}$$

where P_{refl} is the reflected power. Closed-form expressions for the location of both the poles and zeros of power reflection coefficient functions as in (34) for conventional Chebyshev, lumped-element, low-pass filter circuits are also available [13].

By inserting the mapping in (27) into (34), the corresponding power reflection coefficient function

$$\frac{P_{\text{refl}}}{P_{\text{avail}}} = \Gamma(p)\Gamma(-p) = \frac{N(p)N(-p)}{U(p)U(-p)} \tag{35}$$

for the circuit in Fig. 4 is obtained. After throwing away the right-half-plane poles and zeros of this function, and eliminating any cancelling factors that may have been introduced by the mapping process, the voltage reflection coefficient function

$$\Gamma(p) = \frac{N(p)}{U(p)} = \frac{K(p - j\omega_\alpha)(p + j\omega_\alpha)(p - j\omega_\beta)(p + j\omega_\beta)\cdots}{(p - p_1)(p - p_2)(p - p_3)(p - p_4)\cdots} \tag{36}$$

is obtained. It is of interest to note that the denominator of (36) has the same roots as does the numerator of (23), while the numerator of (36) has all of its roots located on the $j\omega$-axis. The voltage reflection coefficient function, (36), is of most immediate use in the synthesis procedure so that the formation of the function in (35) may be bypassed. Thus, only the poles and zeros of (34) which map to the left-half plane or to the $j\omega$-axis are included in the mapping process. The mapping is accomplished most directly by solving (27) for p to obtain

$$p = \pm \sqrt{\frac{j\omega_x'\omega_0^2 A + p'}{-j\omega_x' A + p'}} \cdot \tag{37}$$

Thus, for each pole or zero of (31), (37) will yield two poles and zeros. From this array of mapped poles and zeros, we select all of the poles that are in the left-hand plane, and use them to form (36). Mapping the zeros of (31) will yield double zeros on the $j\omega$ axis of the p-plane, and we select one zero from each of these double zeros to form the numerator of (36).

It is still necessary to specify how the parameter A is to be determined. Now θ_a and θ_b can be determined in terms of θ_m and the fractional bandwidth w by use of

$$\theta_b = \theta_m\left(1 + \frac{w}{2}\right) \tag{38}$$

$$\theta_a = \theta_m\left(1 - \frac{w}{2}\right). \tag{39}$$

By inserting (25) into (26), and using the fact that $\omega' = \omega_x'$ when $\theta = \theta_b$, we may solve for A and obtain

$$A = \frac{1 + \tan^2\theta_b}{\tan^2\theta_b - \tan^2\theta_0} \tag{40}$$

where[2]

$$\omega_0 = \tan\theta_0. \tag{41}$$

Since $\omega' = -\omega_x$ when $\theta = \theta_a$, by use of (25), (26), and (40), we can solve for $\tan\theta_0$ to obtain

$$\tan\theta_0 = \sqrt{\frac{(\tan\theta_b)^2[1+(\tan\theta_a)^2]+(\tan\theta_a)^2[1+(\tan\theta_b)^2]}{2+(\tan\theta_a)^2+(\tan\theta_b)^2}}. \tag{42}$$

Using (42) in (40), we may then compute A.

Other transfer-function parameters may be obtained by mapping from the transfer function for a conventional lumped-element low-pass Chebyshev filter. This transfer function is

$$\frac{P_{\text{avail}}}{P_{\text{out}}} = 1 + \epsilon\cosh^2\left(q\cosh^{-1}\frac{\omega'}{\omega_x'}\right) \tag{43}$$

where

$$\epsilon = \left[\text{antilog}\frac{L_{Ar}}{10} - 1\right]. \tag{44}$$

At the maximum attenuation point in Fig. 2, $\theta = \pi/2$, $\omega = \tan\pi/2 = \infty$, and by (26), $\omega'/\omega_x' = A$. Therefore,

$$\left.\frac{P_{\text{avail}}}{P_{\text{out}}}\right|_{\theta=\pi/2} = 1 + \epsilon\cosh^2\left(\frac{n}{2}\cosh A\right) \tag{45}$$

where the fact that $n = 2q$ has been made use of. The maximum attenuation L_{A1} is this ratio expressed in decibels. Equation (45) will also be useful in determining the constant multiplier K in (36). Since $|\Gamma|$ is equal to K when p equals $j\infty$,

$$|\Gamma|^2 = 1 - \frac{P_{\text{avail}}}{P_{\text{out}}}, \tag{46}$$

and by use of (45) and (46), we obtain

$$|\Gamma|_{p=j\infty} = K = \sqrt{\frac{\epsilon\cosh^2[(n/2)\cosh^{-1}A]}{1 + \epsilon\cosh^2[(n/2)\cosh^{-1}A]}}. \tag{47}$$

[2] Note that θ_0 is not the same as θ_m (see Fig. 2.)

Using these results the reflection coefficient function in (36) can be completely specified.

After the reflection coefficient function is obtained with its numerator and denominator polynomials in factored form, these polynomials must next be multiplied out. Then assuming $Z_0 = 1$, the input impedance function Z_{in} in Fig. 4 is given by

$$Z_{\text{in}} = \frac{U(p) + N(p)}{U(p) - N(p)}. \tag{48}$$

Next the circuit is synthesized by removing one unit element at a time by successive application of (9) of [9]. In this manner the impedance values Z_1, Z_2, etc., associated with the unit elements in Fig. 4, are obtained, and these impedance values are also the characteristic impedances of the line sections in Fig. 1. Since the zeros of the reflection coefficient function in (36) are all on the $j\omega$-axis, the network must be either symmetric or antimetric (in this case it is antimetric) [14], and the second half of the network can be computed from the first half. For this reason it is necessary only to compute half of the impedance values, and the other half can be computed by (8).

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